$$2\sin n = -\int_{-1}^{1} N_2'\,d\xi \tag{8.2}$$

This, together with the assumption that the functions appearing in the above expressions do not undergo large changes, yields

$$N_2' = O(\sin n)$$

But according to our assumption $\sin n$ represents the error of the zero-moment conditions, hence

$$N_2' = O(\eta^{\mu}) \tag{8.3}$$

The fourth equation of equilibrium

$$\frac{\partial H_1}{\partial \xi} - \frac{\partial G_2}{\partial \theta} + rN_2 = 0$$

together with (8.3) and (6.1), yields the following estimate:

$$\sigma_G = O(\eta^{\mu-4})$$

which is in full agreement with (6.2).

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Translated by L. K.

ON BIFURCATION OF THE EQUILIBRIUM MODES OF AN ELASTOPLASTIC ROD AND ANNULUS UNDER PROLONGED LOADING CONDITIONS

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Stability problems of a rectilinear rod and a circular annulus under compression beyond the elastic limit are examined on the basis of the prolonged loading concept.

For an idealized elastoplastic rod model Shanley [1] showed that the least critical value of the axial compressive force is realized under the condition of continuous growth of the external loading during buckling. This result was obtained by static methods and was later expanded by a number of authors [2-5].

Proceeding from the assumption of equilibrium of the deformation process beyond the elastic limit, the stability of a compressed rod is examined taking into account the actual position of the boundary separating the elastic and plastic domains during buckling. By an asymptotic solution of the nonlinear elastoplastic equilibrium equations the character of the branching of the equilibrium modes in the neighborhood of the bifurcation point is investigated.

The bending equations in the post-critical state are obtained by a variational method and generalize the Euler elastic equation to the case of elastoplastic deformations. In conformity with the concept of a prolonged loading, the external force is considered an unknown increasing function of the approach of the rod endpoints. It is assumed that this function admits expansion into a power series.

Utilization of the representation of a variable compressive force in the form of a power series with undetermined coefficients permits linearization of the initial equations and obtaining a parametric equation, in closed form, of the family of curves separating the elastic and plastic zones of the rod.

In conclusion, the stability of elastoplastic equilibrium of a circular annulus subjected to increasing hydrostatic pressure is considered relying on the results obtained in the problem of the compressed rod.

1. Let us examine an ideal hinge-supported rectangular rod, compressed quasi-statically by an increasing force P_1. Beyond the elastic limit the rod material possesses linear hardening.

Let us consider the applied compressive force not to cause shortening of its axis until the beginning of rod twisting at the loading $P_1 = P$.

In calculating the elongation of the fibers a distance z from the middle layer, by limiting ourselves to the linear part of the deformation of the middle fiber, we assume

$$\varepsilon = \frac{du}{ds} - z\frac{d\theta}{ds}, \qquad \sin\theta = \frac{dw}{ds} \tag{1.1}$$

where $u = u(s)$, $w = w(s)$ are the bifurcation displacements, and s is the arclength of the undeformed rod axis.

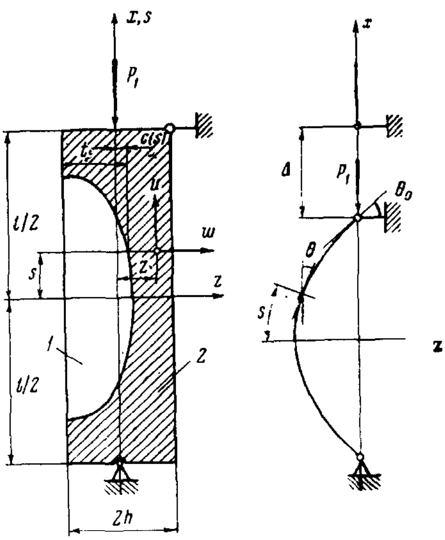

Fig. 1

Because of the passive deformation accompanying buckling beyond the elastic limit, the volume of the rod is separated into elastic 1 and plastic 2 zones (Fig. 1). The function $c(s)$ defines the boundary of the domains of active plastic deformation and unloading. Denoting quantities referring to each of the domains by the superscripts 1 and 2, let us write down the compressive strains and stresses in the postcritical state

$$\varepsilon^{(1)} = -u' + z\theta', \quad \sigma^{(1)} = \frac{P}{4bh} + E_1\varepsilon^{(1)},$$

$$-h \leqslant z \leqslant c(s), \quad \varepsilon^{(2)} = -u' + z\theta', \tag{1.2}$$

$$\sigma^{(2)} = \frac{P}{4bh} + E_2\varepsilon^{(2)}, \quad c(s) \leqslant z \leqslant h$$

The primes in (1.2) denote differentiation with respect to the coordinate s, while $2b, 2h$ are the cross-sectional dimensions and E_1, E_2 are the elastic and tangential moduli of the material.

To obtain the equilibrium equations of an elastoplastic rod under finite bending strains, let us proceed from the principle of minimum total energy in the form taken in deformation plasticity theory. For a body separated into elastic and plastic zones we have [6]

$$\delta Э = \delta (\Pi^{(1)} + \Pi^{(2)}) - \delta A = 0$$

$$\delta\Pi^{(1)} = \int\limits_{v^{(1)}} \sigma^{(1)}\delta\varepsilon^{(1)}\, dv^{(1)} \qquad \delta\Pi^{(2)} = \int\limits_{v^{(2)}} \sigma^{(2)}\delta\varepsilon^{(2)}\, dv^{(2)} \tag{1.3}$$

here $\Pi^{(1)}$, $\Pi^{(2)}$, $dv^{(1)}$, $dv^{(2)}$ are the potentials of the work of deformation and the volume elements of each of the zones, while A is the work of the external forces.

Utilizing (1.1)–(1.3) and the equality $c(s) = u' / \theta'$, which results from the definition of the boundary betaeen the elastic and plastic zones ($\varepsilon = 0,\ \ z = c$), and integrating over the cross-sectional area of the rod, we find

$$\delta (\Pi^{(1)} + \Pi^{(2)}) = \int\limits_{-1/2\, l}^{+1/2\, l} \left[\left(-P + \frac{B}{c} u'\right)\delta u' + K\theta'\delta\theta'\right] ds \tag{1.4}$$

$$B(s) = b\,[E_1 (h + c)^2 - E_2 (h - c)^2]$$

$$K(s) = {}^1/_3 b\,[E_1 (h + c)^2 (2h - c) + E_2 (h - c)^2 (2h + c)]$$

In conformity with the concept of a continuing loading, let us represent the increasing compressive stress resultant P_1 as a power series with undetermined coefficients

$$P_1 = P + \sum_{n=1}^{\infty} b_n \Delta^n \tag{1.5}$$

where $\Delta = \Delta_1 + \Delta_2$, Δ is the total displacement of the upper supports, and Δ_1, Δ_2 are components of the approach of the rod endpoints because of compression and bending during the bifurcation. Let us utilize the formulas [7]

$$\Delta_1 = -\int\limits_{-1/2\, l}^{+1/2\, l} u'\, ds, \qquad \Delta_2 = 2l\left[1 - \frac{E(\varkappa)}{K(\varkappa)}\right], \qquad \varkappa = \sin\frac{\theta_0}{2} \tag{1.6}$$

to calculate the displacements Δ_1, Δ_2 to the accepted accuracy. Here $K(\varkappa)$, $E(\varkappa)$ are the complete elliptic integrals of the first and second kinds with modulus $\varkappa$, and θ_0 is the reference angle of rotation of the rod section.

Introducing the variable $\eta = ks$ from the definition of the functions $K(\varkappa)$, $E(\varkappa)$ taking account of (1.1) and the relationships

$$d\eta' = (1 - \varkappa^2 \sin^2\varphi)^{-1/2} d\varphi = (1 - \varkappa^2 \sin^2\varphi)^{1/2} d\varphi / \cos^2(\theta / 2)$$

$$\sin\varphi = \sin(\theta / 2) / \sin(\theta_0 / 2)$$

we obtain as a result of calculations

$$E(\varkappa) = \int\limits_0^{1/2\,\pi} \sqrt{1 - \varkappa^2 \sin^2\varphi}\, d\varphi = \frac{k}{2} \int\limits_0^{1/2\, l} (1 + \sqrt{1 - w'^2})\, ds \tag{1.7}$$

$$K(\varkappa) = \int\limits_0^{1/2\,\pi} \frac{d\varphi}{\sqrt{1 - \varkappa^2 \sin^2\varphi}} = \frac{kl}{2}$$

where φ is the amplitude of the functions $K(\varkappa)$, $E(\varkappa)$ and k is a parameter.

According to (1.6), (1.7), (1.1), the approach of the rod endpoints under finite bending displacements is of the form

$$\Delta = \Delta_1 + \Delta_2 = l - \int_{-1/2\, l}^{+1/2\, l} (u' + \cos\theta)\, ds \qquad (1.8)$$

Utilizing (1.8) and evaluating the variation in the work of the external forces $\delta A = P_1 \delta\Delta$, we find for $P_1 - P + (B / c)\, u' = Q$ according to (1.3), (1.4)

$$\int_{-1/2\, l}^{+1/2\, l} \left(Q \frac{d}{ds} \delta u + K\theta' \frac{d}{ds} \delta\theta - P_1 \sin\theta\delta\theta\right) ds = 0 \qquad (1.9)$$

Integrating (1.9) by parts, we obtain

$$[Q\delta u + K\theta'\delta\theta]_{-1/2\, l}^{+1/2\, l} - \int_{-1/2\, l}^{+1/2\, l} \left[\frac{dQ}{ds} \delta u + \left(\frac{d}{ds} K\theta' + P_1 \sin\theta\right) \delta\theta\right] ds = 0 \qquad (1.10)$$

The equilibrium conditions of the rod

$$d/ds \left(P_1 - P + \frac{B}{c} u'\right) = 0, \qquad d/ds\,(K\theta') + P_1 \sin\theta = 0 \qquad (1.11)$$

extending the Euler equation to the case of elastoplastic deformations follow from the variational equation (1.10) for independent δu, $\delta\theta$.

To determine the critical value of the external loading, let us linearize the system (1.11).

First let us consider the first of the mentioned equations and let us integrate it for the initial conditions $P_1 = P$, $u = w = 0$, $u' = 0$. Utilizing the expansion (1.5) we have as a result of subsequent linearization

$$P_1 - P + (B / c)\, u' = 0, \qquad \frac{B}{c} u' = b_1 \int_{-1/2\, l}^{+1/2\, l} \left(u' - \frac{1}{2} w'^2\right) ds \qquad (1.12)$$

where b_1 is some undetermined coefficient.

Seeking the approximate solution of the linearized equilibrium equations, and satisfying the prescribed boundary conditions, we set

$$u(s) = -u_0 \left(1 + \sin\frac{\pi s}{l}\right), \qquad w(s) = w_0 \cos\frac{\pi s}{l} \qquad (1.13)$$

From (1.12), (1.13), (1.4) we find

$$\frac{B(s)}{c(s)} \cos\frac{\pi s}{l} = b_1 \left(\frac{2l}{\pi} + \frac{\pi}{4} \frac{w_0^2}{u_0}\right) \qquad (1.14)$$

Because the displacements of a flexible rod u_0 and w_0^2 have the same order of smallness, the ratio w_0^2 / u_0 is a certain constant, and the right side of (1.14) together with the undetermined coefficient b_1 can be combined into a common undetermined parameter

$$\lambda = b_1 \left(\frac{2l}{\pi} + \frac{\pi}{4} \frac{w_0^2}{u_0}\right)$$

Introducing the notation

$$\nu = E_2 / E_1, \quad \alpha = \lambda / E_1 h, \quad c^\circ = c / h, \quad s^\circ = s / l$$

we represent the first equilibrium equation (1.11) in the dimensionless form

$$f(c^\circ, s^\circ, \alpha) = \frac{\alpha c^\circ}{(1 + c^\circ)^2 - \nu (1 - c^\circ)^2} - \cos\pi s^\circ = 0 \qquad (1.15)$$

The transcendental equation (1.15) with parameter α implicitly defines the family of curves separating the elastic and plastic zones of the rod at the instant of buckling.

The character of the boundary curves $c^\circ(s^\circ, \alpha)$ calculated by means of (1.15) for $\nu = {}^1/_2$ is shown in Fig. 2 (solid curves).

Linearizing the second equilibrium equation in (1.11) by using the expansion (1.5), we obtain

$$\frac{d}{ds^\circ}\left(K\frac{d\theta}{ds^\circ}\right) + 4\pi^2\nu P^\circ\theta = 0, \quad P^\circ = \frac{P}{P_*}, \quad P_* = \frac{4\pi^2 bh^3 E_2}{3l^2} \tag{1.16}$$

$$K^\circ = (1 + c^\circ)^2(2 - c^\circ) + \nu(1 - c^\circ)^2(2 + c^\circ)$$

where P_* is the tangent-modulus critical loading.

Setting in conformity with (1.13)

$$\theta = \theta_0 \sin \pi s^\circ, \quad \theta_0 = -\pi w_0 / l \tag{1.17}$$

and applying the procedure of the Bubnov-Galerkin method to (1.16) we find

$$P^\circ = \frac{1}{2\nu} T(\alpha) \tag{1.18}$$

$$T(\alpha) = \int_{-1/2}^{+1/2} K^\circ(s^\circ, \alpha)\cos^2 \pi s^\circ \, ds^\circ$$

It follows from (1.18) that the relative critical stress resultant P° is a function of the undetermined parameter α associated with the disposition of the elastic and plastic zones.

The minimum value of the critical force is realized under the condition of the minimum of the integral (1.18) in the parameter α. Analysis and evaluation by means of (1.18), (1.16), (1.15) show that for a given ratio $\nu = E_2 / E_1$ the function $T(\alpha)$ has a relative minimum at the end of the interval of variation of the parameter $\alpha = \alpha_* = 4\nu$ (Fig. 3).

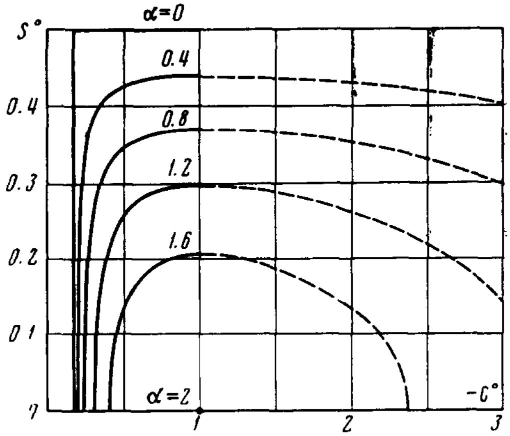

Fig. 2

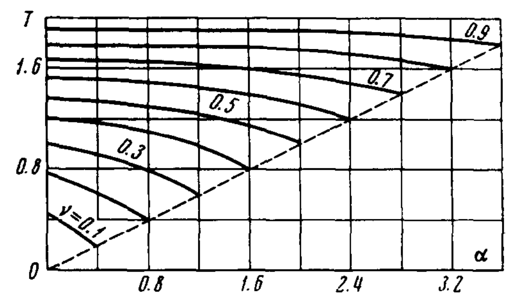

Fig. 3

For $\alpha = \alpha_*$ the equation $c^\circ(s^\circ, \alpha)$ of the family of lines degenerates into a point with the coordinates $s^\circ = 0$, $c^\circ = -1$ (Fig. 2), and (1.16), (1.18) yield

$$K^\circ_* = 4\nu, \quad T(\alpha_*) = 2\nu, \quad P^\circ = 1, \quad P = P_*$$

Therefore, the critical compressive stress resultant of an elastoplastic rod under conditions of continuing loading found by linearizing the equilibrium equations by using the expansion (1.5) corresponds to a tangent-modulus loading.

"Kármán" buckling holds for a constant compressive stress resultant at the instant of buckling. In this case $b_1 = 0$, $\alpha = 0$, $c^\circ = (1 - \sqrt{\nu}) / (1 + \sqrt{\nu})$ and the position of the boundary between the elastic and plastic zones is independent of the rod coordinate s° (Fig. 2).

2. To investigate the post-critical behavior of the rod under compressive stress resultants similar to the tangent-modulus loading, let us turn to the nonlinear equilibrium equations (1.11).

Assuming that the approximation (1.13) is valid as $P_1 \to P_*$ when the displacements u, w are very small, we have from the first equilibrium equation of (1.11)

$$\frac{\alpha_1 c^\circ}{(1+c^\circ)^2 - \nu(1-c^\circ)^2} = \cos \pi s^\circ, \quad \alpha_1 = \frac{l}{\pi b E_1 h}\left(\frac{P_1 - P_*}{u_0}\right) \qquad (2.1)$$

Comparing (2.1) with its corresponding linearized equation (1.15), we find that for small finite deviations of the rod from the rectilinear shape,

$$\alpha = \alpha_1 = \frac{\gamma}{c^\circ(0)}, \quad \gamma = -\frac{4\pi\nu h^\circ (P_1^\circ - 1)}{3\theta_0}, \quad P_1^\circ = \frac{P_1}{P_*} \qquad (2.2)$$

$$c^\circ(0) = -\frac{1}{1-\nu}\left[(1+\nu) - \frac{\alpha_1}{2}\right] + \left\{\frac{1}{(1-\nu)^2}\left[(1+\nu) - \frac{\alpha_1}{2}\right]^2 - 1\right\}^{1/2}$$

should be taken as the parameter of the boundary curves for $u_0 = -hc^\circ(0)\,\theta_0$ and $h^\circ = h/l$.

The parameter α_1, which is undetermined at the instant of buckling under loadings exceeding P_*, connects the position of the elastic and plastic zone boundary with the axis compression and deviation of the rod. The quantity $c^\circ(0)$ is a root of (2.1) for the fixed value of the coordinate $s^\circ = 0$.

Analyzing the post-critical behavior of the rod in the neighborhood of $P_1 = P_*$ and taking into account that for $P_1 \to P_*$ $\alpha_1 \to 4\nu$, $c^\circ(0) \to -1$, $\gamma \to -4\nu$, let us introduce the small parameters

$$\beta = 4\nu - \alpha_1, \quad \omega = 4\nu + \gamma \qquad (2.3)$$

The relationship (2.2) should be considered as an algebraic equation setting up a dependence between the quantities α_1 and γ.

Utilizing (2.2), (2.3) and taking account of the smallness of the quantities ω, β we find in a quadratic approximation

$$(1+3\nu)\beta^2 - 2\left[8\nu^2 + (1-3\nu)\omega\right]\beta + (1-\nu)\omega^2 = 0, \qquad \beta = \frac{1-\nu}{16\nu^2}\omega^2$$

$$\alpha_1 = 4\nu - (1-\nu)\tau^2, \qquad \tau = \frac{\omega}{4\nu} = 1 - \frac{\pi h^\circ (P_1^\circ - 1)}{3\theta_0} \qquad (2.4)$$

To obtain the asymptotic solution of the nonlinear bending equation in the neighborhood of the bifurcation point, let us first investigate the behavior of the family of curves (2.1) with the parameter α_1 as $P_1 \to P_*$. Taking into account that in this case the unloading domain is vanishingly small ($P_1 \to P_*$, $s^\circ \to 0$, $t^\circ = 1 + c^\circ \to 0$) and the approximate formulas

$$\frac{\alpha_1 c^\circ}{(1+c^\circ)^2 - \nu(1-c^\circ)^2} = \frac{4\nu - \beta}{4\nu}\left(1 + \frac{1-\nu}{4\nu}t^{\circ 2} + \ldots\right)$$

$$\cos \pi s^\circ = 1 - \frac{(\pi s^\circ)^2}{2} + \ldots$$

are valid to the accuracy of second order members in the quantities t°, s°, β we obtain

$$\left(\frac{s^\circ}{a^\circ}\right)^2 + \left(\frac{t^\circ}{b^\circ}\right)^2 = 1, \quad a^{\circ 2} = \frac{\beta}{2\pi^2\nu}, \quad b^{\circ 2} = \frac{\beta}{1-\nu}, \quad s_1^\circ = \pm\frac{\tau}{\pi}\left(\frac{1-\nu}{2\nu}\right)^{1/2} \qquad (2.5)$$

here s_1° is the boundary between the elastic and plastic zones on the side surfaces of the rod $(t^\circ = 0)$.

It follows from (2.5) that for compressive forces similar to the tangent-modulus loading the elastic core of the rod is an elliptic domain with semi-axes dependent on axial compression and the amplitudes of the angles of rotation of the cross sections.

According to (1.16), (2.5), (2.4), for

$$c^\circ = t^\circ - 1, \quad g_1 = 4\nu + 3(1-\nu)\tau^2, \quad g_2 = -6\pi^2\nu$$

the dependence of the rod bending stiffness varying along the length, has the following form in a quadratic approximation:

$$(K^\circ)_{P_1 \to P_*} = R^\circ = \begin{cases} 4\nu & (s_1^\circ \leqslant s^\circ \leqslant {}^1/_2) \\ g_1 + g_2 s^{\circ 2} & (0 \leqslant s^\circ \leqslant s_1^\circ) \end{cases} \tag{2.6}$$

Turning to an investigation of the asymptotic solution of the nonlinear bending equation with a stiffness given by (2.6), let us represent (1.11) in the dimensionless form

$$\frac{d}{ds^\circ}\left(R^\circ \frac{d\theta}{ds^\circ}\right) + 4\pi^2\nu P_1^\circ \sin\theta = 0 \tag{2.7}$$

Considering small rod deviations from the rectilinear shape, let us seek the approximate solution of (2.7) by setting $\theta = \theta_0 \sin\pi s^\circ$ in conformity with (1.17). Applying the Bubnov-Galerkin method to the nonlinear equation (2.7) associated with the variational problem (1.3), we have

$$\theta_0 L_1 - 4\pi\nu(\theta_0 L_2 - P_1^\circ L_3) = 0, \quad L_1 = \int_0^{s_1^\circ} \frac{dG_1}{ds^\circ} \sin\pi s^\circ ds^\circ$$

$$L_2 = \int_{s_1^\circ}^{1/2} G_2 \sin\pi s^\circ ds^\circ, \quad L_3 = \int_0^{1/2} G_3 \sin\pi s^\circ ds^\circ$$

$$G_1 = (g_1 + g_2 s^{\circ 2})\cos\pi s^\circ, \quad G_2 = \sin\pi s^\circ, \quad G_3 = \sin(\theta_0 \sin\pi s^\circ)$$

Performing the quadratures

$$L_1 = -\frac{\pi s_1^\circ}{2}\left(g_1 + \frac{g_2 s_1^{\circ 2}}{3}\right) + \frac{1}{4}\left[g_1 + g_2\left(\frac{1}{2\pi^2} + s_1^{\circ 2}\right)\right]\sin 2\pi s_1^\circ - \frac{g_2 s_1^\circ}{4\pi}\cos 2\pi s_1^\circ$$

$$L_2 = {}^1/_4 - {}^1/_2 s_1^\circ + {}^1/_4 \pi^{-1}\sin 2\pi s_1^\circ, \quad L_3 = {}^1/_2 I_1(\theta_0)$$

and taking account of the smallness of the quantities s_1°, we arrive at a transcendental equation connecting the angle of rotation with the axial compression of the rod

$$1 + 5\pi^2 s_1^{\circ 3} + 2P_1 I_1(\theta_0)/\theta_0 = 0$$

where $I_1(\theta_0)$ is the Bessel function of the first kind.

Utilizing the approximation $I_1(\theta_0) \approx {}^1/_2\theta_0$ for $\theta_0 \ll 1$ and the relationships (2.4), (2.5), we find

$$\theta_0 = \frac{{}^1/_3 \pi h^\circ (P_1^\circ - 1)}{1 - m(P_1^\circ - 1)^{1/2}}, \qquad m = \left[\frac{2\pi\nu}{5(1-\nu)}\left(\frac{2\nu}{1-\nu}\right)^{1/2}\right]^{1/3}$$

Determining the slope of the curve $\theta_0 = \theta_0(P_1^\circ)$ at the point $P_1 = P_*$ and going over to the amplitude of the rod deflections in conformity with (1.17), we obtain for $w^\circ = -w_0/h$

$$w^\circ = \frac{{}^1/_3(P_1^\circ - 1)}{1 - m(P_1^\circ - 1)^{1/2}}, \qquad \left(\frac{dw^\circ}{dP_1^\circ}\right)_* = \frac{1}{3} \tag{2.8}$$

It follows from (2.8) that the point with coordinates $P_1^\circ = 1$, $w^\circ = 0$ is a branch point of the equilibrium mode of a compressed elastoplastic rod (Fig. 4). The slope of the tangent to the curve $w^\circ = w^\circ(P_1^\circ)$ at $P_1^\circ = 1$ agrees with the results in [3, 5].

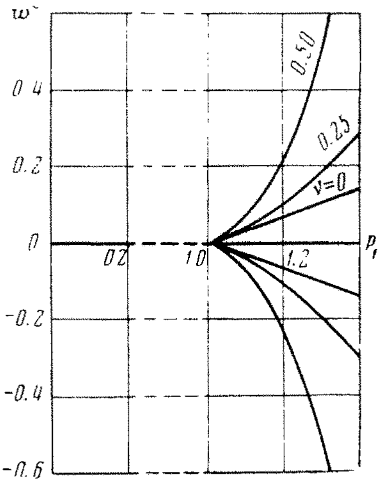

Fig. 4

3. Utilizing the results of Sect. 1, and keeping the same notation, unless specified otherwise, let us examine a kindred problem of the stability of the elastoplastic equilibrium of a circular ring of rectangular cross section subjected to increasing hydrostatic pressure.

The solution of the mentioned problem results in a system of nonlinear bending differential equations

$$\frac{d}{ds}\left(\frac{B}{c}\varepsilon\right)-\left(\frac{1}{R}-\theta'\right)\frac{d}{ds}(K\theta')=0$$

$$\frac{d^3}{ds^3}(K\theta')+\frac{1}{R}\left(\frac{1}{R}-\theta'\right)\frac{d}{ds}(K\theta')-$$

$$-\frac{d}{ds}(B\theta'^2)+qR\theta''=0 \qquad (3.1)$$

resulting from the equilibrium conditions of a deformed ring element

$$\frac{Q}{R}-Q\theta'+\frac{dN}{ds}=0, \quad \frac{N}{R}-N\theta'-\frac{dQ}{ds}=-q_1, \quad Q=\frac{dM}{ds}$$

Here ε, θ are the finite deformation components

$$\varepsilon = e + {}^1/_2(e^2+\gamma^2)+z\varkappa, \qquad \varkappa=-\theta', \qquad \sin\theta=\gamma$$
$$\cos\theta = 1+e, \qquad \gamma = w' - v/R, \qquad e = v' + w/R$$

where v, w are the ring displacement in the tangential and outer normal directions, R is the radius, q, q_1 the linear hydrostatic loading at the instant of bifurcation and in the post-critical state, and N, Q, M are the axial stress resultant, the transverse force, and the bending moment, respectively. Relationships between the stress resultants and strains found by integrating the stresses (1.2) over the cross-sectional area for $P=-qR$, $\varepsilon^{(1)} = \varepsilon^{(2)} = e + {}^1/_2(e^2+\gamma^2) - z\,\theta'$ are

$$N=-qR+\frac{B(s)}{c}\varepsilon, \quad M=-K(s)\theta' \quad (-{}^1/_4\pi R \leqslant s \leqslant +{}^1/_4\pi R)$$

Here and henceforth, we shall consider buckling of the ring with the formation of two half-waves. Such an alternation of the elastic and plastic zones in the circumferential direction, for which the arc $-{}^1/_4\pi R \leqslant s \leqslant +{}^1/_4\pi R$ might be considered instead of the closed ring because of symmetry, will correspond to this case.

Linearizing the bending equation (3.1) and discarding the member with the factor $(h/R)^2$ in the first of the equations for a thin circular ring, we find

$$\frac{d}{ds}\left\{\frac{1}{c^\circ}\left[(1+c^\circ)^2-\nu(1-c^\circ)^2\right]\left(v'+\frac{w}{R}\right)\right\}=0, \qquad q^\circ=\frac{q}{q^*} \qquad (3.2)$$

$$\frac{d}{ds}\left[R^2\frac{d^2}{ds^2}(K^\circ\theta')+(12\nu q^\circ+K^\circ)\gamma'\right]=0, \qquad q^*=\frac{4bh^3E_2}{R^3}$$

Integrating the first equation of the system (3.2), we have

$$F(c^\circ)\left(v' + \frac{w}{R}\right) = C_1 = \text{const}, \quad F(c^\circ) = \frac{1}{c^\circ}[(1+c^\circ)^2 - \nu(1-c^\circ)^2] \quad (3.3)$$

Proceeding form the concept of a continuing loading, we seek the solution of the differential equation (3.3) as

$$v = v_0 \sin\frac{2s}{R} + v_*(s), \qquad w = w_0 \cos\frac{2s}{R} + w_*(s) \qquad (3.4)$$

taking account of the bending and multilateral compression displacement during bifurcation. Here v_*, w_* are the additional displacements of the elastoplastic ring under uniform external pressure as a curved variable-stiffness rod.

Utilizing (3.3), (3.4) and setting $(v_*' + w_* / R) = e_*(s) = -e_0 / F(c^\circ)$ within the limits of the considered quarter ring, we obtain

$$F(c^\circ)(2v_0 + w_0)\cos\frac{2s}{R} = (C_1 + e_0)R$$

Evaluating the constant of integration C_1 from the conditions $v_0 = w_0 = 0$ and $e_0 = 0$ which are valid at the instant preceding bifurcation, and introducing the undetermined parameter $\alpha = e_0 R / (2v_0 + w_0)$, we arrive at the expression

$$F(c^\circ)\cos s^\circ = \alpha, \quad s^\circ = 2s / R,$$
$$-\pi / 2 \leqslant s^\circ \leqslant +\pi / 2 \qquad (3.5)$$

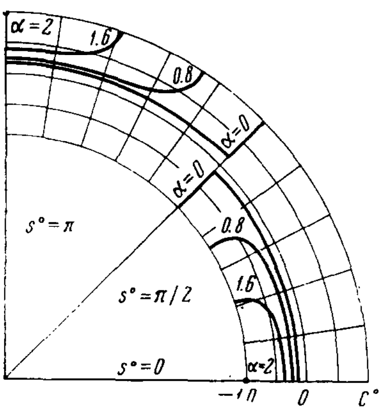

Fig. 5

The transcendental equation (3.5) determines the position of the boundary between the elastic and plastic zones to the accuracy of the parameter α and agrees with the corresponding equation for a compressed rod. Presented in Fig. 5 is the character of the curves $c^\circ(s^\circ, \alpha)$ constructed by means of this equation.

To find the critical value of the multilateral external pressure, let us use the Bubnov-Galerkin method. Seeking the solution of the second equation of the system (3.2) in the form $\gamma = \gamma_0 \sin s^\circ$, and performing the quadratures, we find

$$q^\circ = \frac{1}{2\nu} T(\alpha) \qquad (3.6)$$

where $T(\alpha)$ is a function of the parameter α defined by (1.18).

Utilizing the results of solving the problem on rod stability, we conclude that the minimal critical pressure of a circular ring is realized for $\alpha = \alpha_* = 4\nu$, $K^\circ = K_*^\circ = 2\nu$, $T(\alpha_*) = 2\nu$ and corresponds to the tangent-modulus loading

$$q^\circ = 1, \quad q = q_*, \quad \lambda = (3E_2 / \sigma_*)^{1/2}, \quad \lambda = R / i$$

here σ_* is the critical compressive stress, λ the flexural stiffness, i the radius of inertia of a ring section.

To obtain the solution of the problem of "Kármán" buckling of an elastoplastic ring, the circumferential compressive strain should be put equal to zero. For $e^\circ = 0$ and $\alpha = 0$, the width of the plastic zones within the limits of each quarter ring turns out to be constant (Fig. 5), and the procedure of calculation by using (3.5), (3.6) results in a critical external pressure corresponding to a reduced-modulus loading.

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ON THE THREE-DIMENSIONAL PROBLEM OF MAGNETOELASTIC PLATE VIBRATIONS

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The problem of investigating the magnetoelastic vibrations of an electrically conducting plate in a magnetic field reduces to the combined solution of the magnetoelasticity equations in the domain occupied by the plate (interior problem), and the electrodynamics equations of the rest of the domain of the space under consideration (exterior problem).

An attempt is made to determine the magnetic field of a thin plate of finite conductivity by the asymptotic integration of the combined equations of magnetoelasticity for the domain occupied by the plate. Jointly considering the exterior and interior problems, the magnetoelastic vibrations of a thin plate of finite conductivity are investigated. Some magnetoelasticity hypotheses are formulated for a plate of finite conductivity.

In particular cases when the plate material is ideally conductive or a thin plate of infinite extent has finite electrical conductivity, the problem of the magnetoelastic vibrations is solved relatively simply [1, 2].

In the general case when the plate can have finite dimensions, and its material is finitely conductive, the solution of the problem posed becomes quite difficult, because the interior problem in this case does not separate, and the exact determination of the magnetic field of the plate in a three-dimensional formulation is not possible.

1. An isotropic elastic plate of constant thickness $2h$ fabricated from a material with finite electrical conductivity and in an external magnetic field with given intensity vector $\mathbf{H}_0$ (H_1, H_2, H_3) is considered.

It is assumed that the magnetic and dielectric permeability of the plate equal one.

The Maxwell equations for a vacuum [3] are considered valid for the exterior domain (for the whole domain outside the plate).